Mathematics of the Future: Understanding and Application of Mathematics with the help of Technology, FutureMath Ref. No.: 2020-1-RS01-KA203-065388

Co-funded by the Erasmus+ Programme of the European Union

Improper integrals

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January 2022

Outline

Improper integrals

We consider the function $f(x) = \frac{1}{x^2}, x > 0$. We wish to compute the area of the domain delimited by the plot of the function f, the Ox axis and the line $x = 1$. The area of the region S located to the left of the line $x = u$ is:

$$A(u) = \int_1^u \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^u = 1 - \frac{1}{u}.$$

Improper integrals

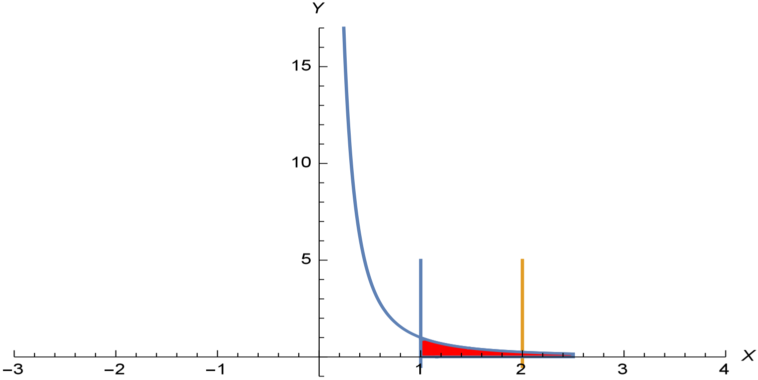

Fig. 1

Improper integrals

Remarks:

- $A(u) \leq 1$, for any value of u;
- $\lim_{u\to\infty} A(u) = \lim_{u\to\infty}(1 - \frac{1}{u}) = 1$ hence

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{u\to\infty}\int_1^u \frac{1}{x^2}dx = 1.$$

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Improper integrals

The conceptul of Riemann *integrability* can be generalized to both functions defined on unbounded intervals and unbounded functions:

- $f : [a, b) \to \mathbb{R}$ where $b = +\infty$, i.e. $[a, b)$ is an *unbounded* interval, or b finite but the function f is *unbounded*
- $f : (a, b] \to \mathbb{R}$ where $a = -\infty$, i.e. $[a, b)$ is an *unbounded* interval, or a finite but the function f is *unbounded*.

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Improper integrals

Remark The following results will be presented only for $f : [a, b) \to \mathbb{R}$ because they can be easily reformulated for $f : (a, b] \to \mathbb{R}$ by means of the simple change of variables $x \to -x$ in the case of an unbounded interval, or by menas of simple change of variables $x \to a + b - x$ in the case of an unbounded function.

Let $f : [a, b) \to \mathbb{R}$ be a function which is integrable on any interval $[a, u] \subset [a, b)$. In this case it makes sense to consider the function:

$$F : [a, b) \to \mathbb{R}, \quad F(u) \stackrel{def}{=} \int_a^u f(x)dx.$$

Improper integrals

Example 1. Study the convergence of the integral $\int\limits_1^\infty \frac{1}{x}dx$.

Solution: $\int\limits_1^\infty \frac{1}{x}dx = \lim\limits_{u\to\infty} \int\limits_1^u \frac{1}{x}dx = \lim\limits_{u\to\infty} \ln x \Big|_1^u = \lim\limits_{u\to\infty} (\ln u - \ln 1) = \infty$. From the definition it folows that the integral is divergent. Figure 2 presents the plot of the function.

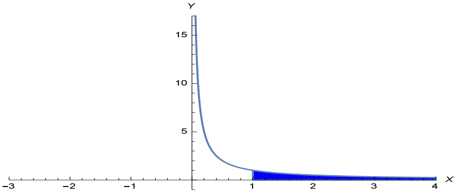

Fig. 2

Improper integrals

Definition of an improper Integral of Type I

- The function $f : [a, \infty) \to \mathbb{R}$ integrable on any interval $[a, u] \subset [a, \infty)$ is called *integrable* in a generalized sense on $[a, \infty)$ if and only if the limit $\lim\limits_{u\to\infty} \int_a^u f(x)dx$ exists and is finite. In this case we call the *improper integral* $\int_a^\infty f(x)dx$ **convergent** and we define:

$$\boxed{\int_a^\infty f(x)dx \stackrel{def}{=} \lim_{u\to\infty} \int_a^u f(x)dx.}$$

Improper integrals

Definition of an improper Integral of Type I

- The function $f : (-\infty, b] \to \mathbb{R}$ integrable on any interval $[u, b] \subset (-\infty, b]$ is called *integrable* in a generalized sense on $(-\infty, b]$ if and only if the limit $\lim\limits_{u \to -\infty} \int_u^b f(x)dx$ exists and is finite. In this case we call the *improper integral* $\int_{-\infty}^b f(x)dx$ **convergent** and we define:

$$\boxed{\int_{-\infty}^b f(x)dx \overset{def}{=} \lim_{u \to -\infty} \int_u^b f(x)dx.}$$

Remark: If the above limits do not exist or if their values are infinite, the integrals are called **divergent**.

Improper integrals

Definition of an improper Integral of Type II

- The function $f:[a,b)\to\mathbb{R}$ integrable on any interval $[a,u]\subset[a,b)$ is called *integrable in a generalized sense* on $[a,b)$ if and only if the limit $\lim\limits_{u\to b, u<b}\int_a^u f(x)dx$ exists and is finite. In this case we call the *improper integral* $\int_a^b f(x)dx$ **convergent** and we define:

$$\boxed{\int_a^b f(x)dx \overset{def}{=} \lim_{u\to b, u<b}\int_a^u f(x)dx.}$$

If the above limit do not exist or if its value is infinite, the integral $\int_a^b f(x)dx$ is called divergent.

Improper integrals

Definition of an improper Integral of Type II

- The function $f : (a, b] \to \mathbb{R}$ integrable on any interval $[u, b] \subset (a, b]$ is called *integrable* in a generalized sense on $(a, b]$ if and only if the limit $\lim\limits_{u \to a, u > a} \int_u^b f(x)dx$ exists and is finite. In this case we call the *improper integral* $\int\limits_a^b f(x)dx$ **convergent** and we define:

$$\boxed{\int_a^b f(x)dx \stackrel{def}{=} \lim_{u \to a, u > a} \int_u^b f(x)dx.}$$

If the above limit do not exist or if its value is infinite, the integral $\int_a^b f(x)dx$ is called divergent.

Improper integrals

Definition of an improper Integral of Type II

- If the function is discontinuous in c, where $c \in (a, b)$, but both integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent then:

$$\boxed{\int_a^b f(x)dx \stackrel{def}{=} \int_a^c f(x)dx + \int_c^b f(x)dx.}$$

Improper integrals

Example 2. Calculate $\int_0^3 \frac{dx}{x-1}$.

Solution: $\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$.

We have $\int_0^1 \frac{dx}{x-1} = \lim\limits_{u\to 1, u<1} \int_0^u \frac{dx}{x-1} = \lim\limits_{u\to 1, u<1} \ln|x-1| \Big|_0^u = \lim\limits_{u\to 1, u<1} \ln|u-1| - \ln|-1| = -\infty$, so there is no point in calculating the second integral since it follows that the initial integral is divergent.

Caution! It would be wrong to calculate $\int_0^3 \frac{dx}{x-1} = \ln|x-1| \Big|_0^3 = \ln 2 - \ln 1$ because the integral is an improper one and it must be calculated using the corresponding method.

Remarkable improper integrals

- $\int_0^{\infty} e^{-mx} dx = \frac{1}{m}, \quad m > 0;$
- $\int_a^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{convergent}, & p \in (1, \infty) \\ \text{divergent}, & p \in (0, 1] \end{cases}, a > 0.$
  Moreover, if $p \in (1, \infty)$ then $\int_a^{\infty} \frac{1}{x^p} dx = \frac{1}{(p-1)a^{p-1}}$
- $\int_a^{b-0} \frac{1}{(b-x)^p} dx = \begin{cases} \text{convergent}, & p \in (0, 1) \\ \text{divergent}, & p \in [1, \infty) \end{cases}, a < b.$
  Moreover, if $p \in (0, 1)$ then $\int_a^{b-0} \frac{1}{(b-x)^p} dx = \frac{(b-a)^{1-p}}{1-p}.$

Convergence tests

Comparison test

Let $f, g : [a, b) \to \mathbb{R}_+$ be two functions integrable on $[a, u] \subset [a, b)$, $\forall u \in (a, b)$. If $f(x) \leq g(x)$, $\forall x \in [a, b)$ then:

- If $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)dx$ is convergent;
- If $\int_a^b f(x)dx$ is divergent then $\int_a^b g(x)dx$ is divergent.

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Convergence tests

Practical convergence test

- Let $f:[a,\infty)\to\mathbb{R}_+$ be integrable on $[a,u]\subset[a,\infty)$ such that $\lim\limits_{x\to\infty} x^p f(x)$ is finite and nonzero for $p\in(1,\infty)$. Then $\int_a^\infty f(x)dx$ is convergent.
- Let $f:[a,b)\to\mathbb{R}_+$ be integrable on $[a,u]\subset[a,b)$ such that $\lim\limits_{x\to b,x<b}(b-x)^p f(x)$ is finite and nonzero for $p\in(0,1)$. Then $\int_a^{b-0} f(x)dx$ is convergent.
- Let $f:(a,b]\to\mathbb{R}_+$ be integrable on $[a,u]\subset(a,b]$ such that $\lim\limits_{x\to a,a<x}(x-a)^p f(x)$ is finite and nonzero for $p\in(0,1)$. Then $\int_{a+0}^{b} f(x)dx$ is convergent.

Convergence tests

Practical convergence test

- Let $f : [a, \infty) \to \mathbb{R}_+$ be integrable on $[a, u] \subset [a, \infty)$ such that $\lim\limits_{x\to\infty} x^p f(x)$ is finite and nonzero for $p \in (1, \infty)$. Then $\int_a^\infty f(x)dx$ is convergent.
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- Let $f : (a, b] \to \mathbb{R}_+$ be integrable on $[a, u] \subset (a, b]$ such that $\lim\limits_{x\to a, a<x} (x - a)^p f(x)$ is finite and nonzero for $p \in (0, 1)$. Then $\int_{a+0}^b f(x)dx$ is convergent.

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Practical convergence test

- Let $f : [a, \infty) \to \mathbb{R}_+$ be integrable on $[a, u] \subset [a, \infty)$ şuch that $\lim\limits_{x\to\infty} x^p f(x)$ is finite and nonzero for $p \in (1, \infty)$. Then $\int_a^\infty f(x)dx$ is convergent.
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Convergence tests

Remark: The conclusions regarding the convergence remain the same even if the limit from the hypothesis is zero. However, if we use the practical test to establish the **divergence** of an improper integral (i.e. if $p \in [0, 1]$ in the first case or if $p \in [1, \infty)$ in the other two cases) then the limit must be nonzero.

Example

Example 3. Study the convergence of the integral: $\int_1^\infty \frac{\arctan(x^7)}{x\sqrt{x}-1}dx$.

Solution: Since 1 and ∞ are the singular points we have:

$$I=\int_1^\infty \frac{\arctan(x^7)}{x\sqrt{x}-1}dx=\int_1^2 \frac{\arctan(x^7)}{x\sqrt{x}-1}dx+\int_2^\infty \frac{\arctan(x^7)}{x\sqrt{x}-1}dx.$$

For the first integral we have: $\lim\limits_{x\to 1, x>1}(x-1)^p\frac{\arctan(x^7)}{x\sqrt{x}-1}=$

$$=\frac{\pi}{4}\lim_{x\to 1, x>1}(x-1)^p\frac{1}{\sqrt{x^3}-1}=\frac{\pi}{4}\lim_{x\to 1, x>1}(x-1)^p\frac{\sqrt{x^3}+1}{(x-1)(x^2+x+1)}=\frac{\pi}{6}$$

for $p=1$ which means that from the practical convergence test it follows that the integral $I_1=\int_1^2 \frac{\arctan(x^7)}{x\sqrt{x}-1}dx$ is divergent.

Example

For the second integral $I_2 = \int_2^{\infty} \frac{\arctan(x^7)}{x\sqrt{x}-1}dx$ we have:

$$\lim_{x\to\infty} x^p \frac{\arctan(x^7)}{x\sqrt{x}-1} = \frac{\pi}{2} \lim_{x\to\infty} \frac{x^p}{x^{\frac{3}{2}}-1} = \frac{\pi}{2}$$

for $p = \frac{3}{2} > 1$ hence the integral $\int_2^{\infty} \frac{\arctan(x^7)}{x\sqrt{x}-1}dx$ is convergent.
Since I_1 is divergent and I_2 is convergent we conclude that $I = I_1 + I_2$ is divergent.

Example

Example 4.

The next example is based on a real-life situation. We suppose that at a very busy traffic crossroad an accident used to occur in average every three months. In an attempt to solve the problem, some changes were implemented in the traffic lights systems and since then, in the last 8 months, no accidents occurred. Now the question is: are these accident-free 8 months the result of the traffic light changes or can they be attributed to pure chance?

The probability theory tells us that if k is the average time between events, then the probability for X, the time between events, to have a value between a and b, is given by $P(a \leq X \leq b) = \int_a^b f(x)dx$, where $f(x) = \begin{cases} 0, & x < 0 \\ ke^{-kx}, & x \geq 0 \end{cases}$.

Example

Thus, if the accidents occur in average once every three months then the probability for X, the time between events, to have a value between a and b, is given by $P(a \leq X \leq b) = \int_a^b f(x)dx$, where $f(x) = \begin{cases} 0, & x < 0 \\ 3e^{-3x}, & x \geq 0 \end{cases}$. To answer the above question we must compute

$$P(X \geq 8) = \int_8^{\infty} 3e^{-3x} dx$$

and to asses if it is possible for 8 months to pass without any accidents taking place in the case when no changes are implemented in the traffic lights systems.

Example

Soluţie. We can calculate this probability as an improper integral:

$$P(X \geq 8) = \int_8^\infty 3e^{-3x}dx = \lim_{u\to\infty}\int_8^u 3e^{-3x}dx$$

$$= \lim_{u\to\infty}(-e^{-3u} + e^{-24}) = e^{-24} \simeq 3,8 \times 10^{-11}.$$

The value $3,8 \times 10^{-11}$ represents the probability that no accidents happen in 8 months if no changes are made to the traffic lights systems. Since this value is very very low, almost zero, it is reasonable to conclude that the changes implemented in the traffic lights systems were efficient indeed.

Homework

Study the convergence to the following integrals and if the integral is convergent, calculate its value:

1) $\int_{-3}^{4} \frac{3x}{\sqrt{x+3}}dx$; 2) $\int_{-2}^{5} \frac{2x-1}{\sqrt{x+2}}dx$; 3) $\int_{1}^{5} \frac{x}{\sqrt{5-x}}dx$;

4) $\int_{0}^{\infty} \frac{2x+1}{(x^2+1)(x+1)}dx$; 5) $\int_{0}^{\infty} \frac{2x-1}{(x^2+1)(x+1)}dx$; 6) $\int_{0}^{\infty} \frac{2x-1}{(x+1)^3}dx$;

7) $\int_{1}^{\infty} \frac{1}{x(x^6+1)}dx$; 8) $\int_{-\infty}^{-2} \frac{2x+1}{(x^2+1)(x+1)}dx$; 9) $\int_{-\infty}^{0} \frac{1}{(x^2+1)(x^3-1)}dx$;

10) $\int_{0}^{\infty} \frac{8x+1}{(x^2+9)(x+1)}dx$; 11) $\int_{1}^{3} \frac{1}{x\sqrt{3-x}}dx$; 12) $\int_{0}^{\infty} \frac{2x+1}{(x^2+1)(x+1)}dx$;

Homework

13) $\int_{-\infty}^{4} \frac{1}{x^2-5x+7}dx$; 14) $\int_{4}^{\infty} \frac{x-1}{x(x+1)(x-3)}dx$; 15) $\int_{1}^{5} \frac{1}{\sqrt[5]{5-x}}dx$;

16) $\int_{1}^{\infty} \frac{1}{x(x^2+4)}dx$; 17) $\int_{1}^{2} \frac{2x+4}{\sqrt[3]{x-1}}dx$; 18) $\int_{2}^{3} \frac{3x-1}{\sqrt[5]{3-x}}dx$;

19) $\int_{3}^{4} \frac{x-1}{\sqrt[4]{(4-x)^3}}dx$; 20) $\int_{1}^{\infty} \frac{1}{\sqrt{x^2+1}}dx$; 21) $\int_{0}^{1} \frac{\sin x}{1-x^2}dx$;

22) $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$; 23) $\int_{1}^{\infty} \frac{x}{\sqrt{x^3-1}}dx$; 24) $\int_{1}^{\infty} \frac{1}{x\sqrt{x}-1}dx$;

25) $\int_{1}^{\infty} \frac{1}{x\sqrt{x^{10}+x^5+1}}dx$.

Homework

Answers: 1) $-4\sqrt{7}$, 2) $-\frac{2\sqrt{7}}{3}$, 3) $\frac{44}{3}$, 4) $\frac{3\pi}{4}$ 5) $\frac{\pi}{4}$ 6) $\frac{1}{2}$, 7) $\frac{\ln 2}{6}$,
8) $\frac{3\pi}{4}-\frac{3}{2}\arctan 2+\frac{\ln 5}{4}$, 9) $-\frac{2\pi\sqrt{3}}{9}$, 10) $\frac{7\pi-3\ln 3}{15}$, 11) $\frac{\ln(5+2\sqrt{6})}{\sqrt{3}}$,
12) $\frac{1}{3\ln\frac{5}{2}}$, 13) $\frac{5\pi\sqrt{3}}{9}$, 14) $\frac{1}{6}\ln\frac{125}{16}$, 15) $\frac{5\sqrt[5]{256}}{4}$, 16) $\frac{\ln 5}{8}$, 17) $\frac{51}{5}$,
18) $\frac{25}{3}$, 19) $\frac{56}{5}$, 20) divergent, 21) divergent, 22) π, 23) divergent,
24) divergent, 25) $\frac{1}{5}\ln\frac{3+2\sqrt{3}}{3}$.

References

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- D. Păunescu, A. Juratoni, *Advanced Integral Calculus*, Ed. Orizonturi Universitare (2022).